Grade 11/12 Math Circles
February 28, March 6, 2024
Population Modeling - Solutions

Exercise Solutions

Exercise 1

Suppose there were 100 rabbits in a region back in 2020. If the population grows by 5% every month, how many rabbits would there be in 2023?

Exercise 1 Solution

We set 2020 as our initial time point, $t = 0$. So $N(0) = 100$. We also know that monthly growth rate $r_d = 0.05$. So we have

$$N(3) = (1 + r_d)^{3\times 12} N_0 = (1 + 0.05)^{36}(100) \approx 579 \text{ rabbits.}$$

Exercise 2

Using the diagrams on the Appendix page, draw the cobweb diagrams for $N(t+1) = pN(t)(1 - N(t))$ with the following values of p and $N_0 = 0.1$. Find where each trajectory ends up. Discuss what trajectory each cobweb diagram stands for.

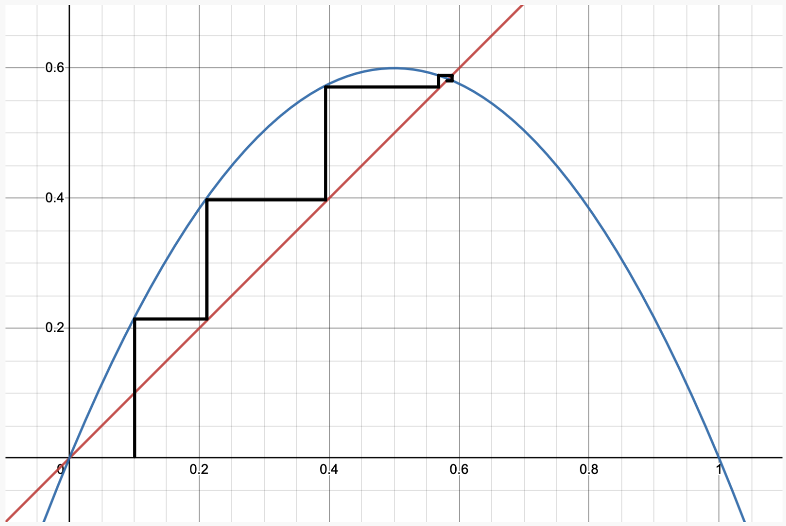
(a) $p = 2.4$

(b) $p = 3.2$

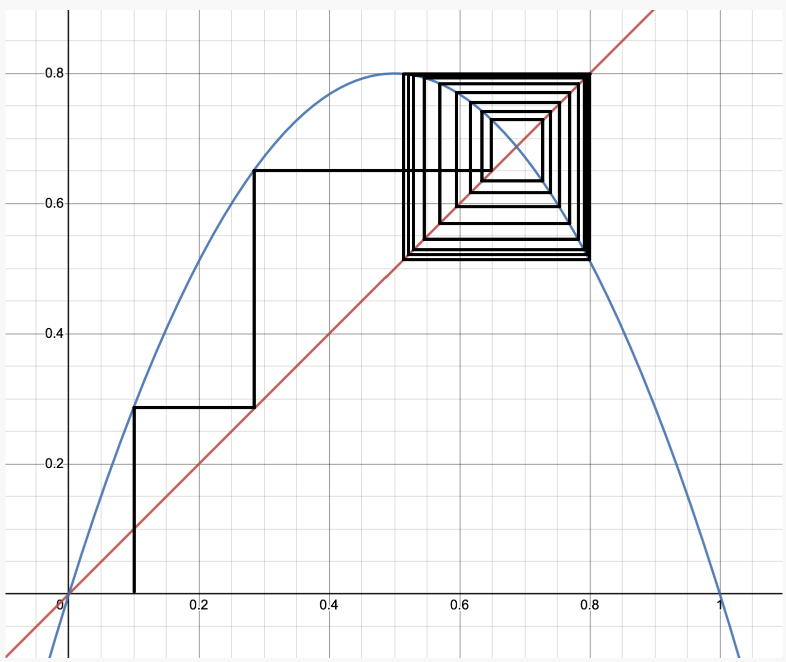
(c) $p = 4.0$

Exercise 2 Solution

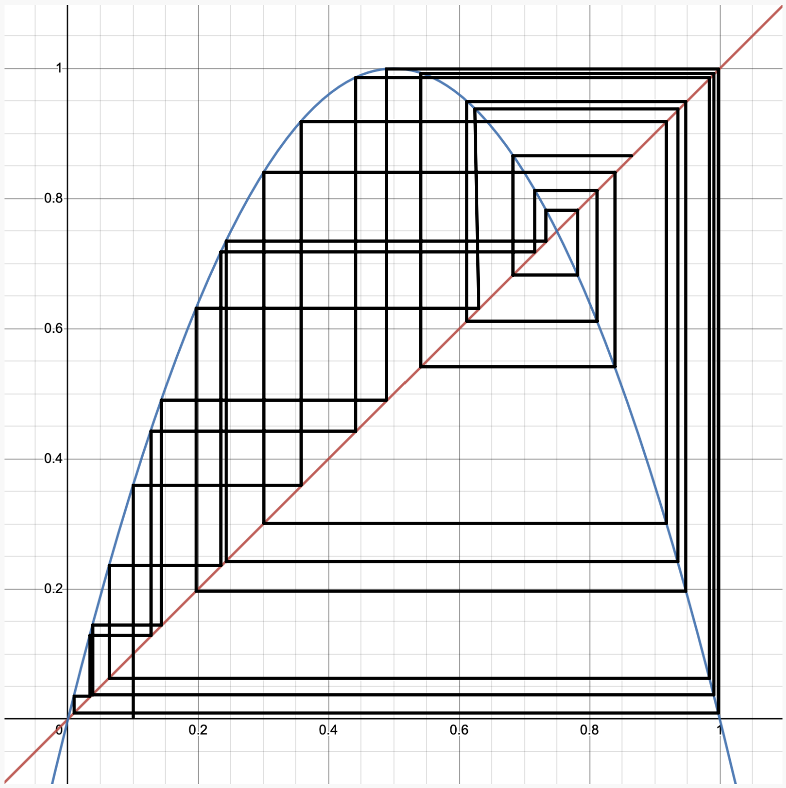
(a) When $p = 2.4$, we see that the trajectory approaches the fixed point in an oscillatory way. This corresponds to the situation when the population overshoots the carrying capacity and exhibits damped oscillations.



(b) When $p = 3.2$, we see that there are two points $\bar{N}_1$ and $\bar{N}_2$ such that $\bar{N}_2 = f(\bar{N}_1)$ and $\bar{N}_1 = f(\bar{N}_2)$. This corresponds to the situation when the population eventually oscillates between 2 values (2-cycle).



(c) When $p = 4.0$, with $N_0 = 0.1$, we obtain a chaotic trajectory!



Exercise 3

Scientists found that there is usually a minimal threshold value for the species to survive. Such a phenomenon, called "Allee Effect", changes the logistic model to

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{A} - 1\right).$$

Analyze the equation by considering

(a) what will happen to $N(t)$ if $A < N_0 < K$?

(b) what will happen to $N(t)$ if $0 < N_0 < A$?

Exercise 3 Solution

(a) If $A < N_0 < K$, we have $N(0) > 0$, $1 - \dfrac{N(0)}{K} > 0$, and $\dfrac{N(0)}{A} - 1 > 0$, so $\dfrac{dN}{dt} > 0$ at $t = 0$, which means that the population will increase.

(b) If $0 < N_0 < A$, we have $N(0) > 0$, $1 - \dfrac{N(0)}{K} > 0$, and $\dfrac{N(0)}{A} - 1 < 0$, so $\dfrac{dN}{dt} < 0$ at $t = 0$, which means that the population will decrease. This tells us that when the population is below A, even if it is below the carrying capacity K, the population is dying out.

Exercise 4

Below is a life table that purely depends on age. Assume we know that the average life span of an animal is 4 years.

Age	0	1	2	3	4
0	0	0	5	10	50
1	0.01	0	0	0	0
2	0	0.25	0	0	0
3	0	0	0.25	0	0
4	0	0	0	0.4	0

(a) At what age does this species start to give birth to babies?

(b) Explain why the entries below the sub-diagonal (bottom left) are all zeros.

(c) Explain why the diagonal is all zeros.

Exercise 4 Solution

(a) Age 2. Note that $a_{12} = 0$, so 1-year-old individuals cannot give birth to babies.

(b) The age will grow by 1 after a year. For example, a 1-year-old individual will be 2 years old after a year, it cannot be 3 or 4 years old.

(c) Since the age will grow by 1 after a year, individuals will not be the same age in the next year.

Exercise 5

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

calculate

(a) AB

(b) BA

(c) AD

(d) CA

Exercise 5 Solution

(a)

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

(b)

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 \times 1 + 6 \times 3 & 5 \times 2 + 6 \times 4 \\ 7 \times 1 + 8 \times 3 & 7 \times 2 + 8 \times 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

(c)

$$AD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 2 \\ 3 \times 1 + 4 \times 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

(d)

$$CA = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 3 & 1 \times 2 + 2 \times 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \end{bmatrix}$$

Exercise 6

The matrix

$$A = \begin{bmatrix} 0 & 50 & 300 \\ 0.02 & 0.25 & 0 \\ 0 & 0.08 & 0.5 \end{bmatrix}$$

can be written as the sum of two matrices, $A = R + S$, where R only contains reproduction rates and S only contains survival rates. Write down R and S, use the same initial population matrix

$$N_0 = \begin{bmatrix} 1000 \\ 50 \\ 10 \end{bmatrix}$$

and do the following:

(a) Calculate RN_0 and explain what the result represents.

(b) Calculate SN_0 and explain what the result represents.

(c) Sum up RN_0 and SN_0 and compare with AN_0. What do you find?

Exercise 6 Solution

We have

$$R = \begin{bmatrix} 0 & 50 & 300 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; S = \begin{bmatrix} 0 & 0 & 0 \\ 0.02 & 0.25 & 0 \\ 0 & 0.08 & 0.5 \end{bmatrix}.$$

(a)

$$RN_0 = \begin{bmatrix} 0 & 50 & 300 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 + 2500 + 3000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5500 \\ 0 \\ 0 \end{bmatrix}$$

This is the number of newly produced individuals after 1 year. Juveniles and adults are not newly born so the number is 0 for both stages.

(b)

$$RN_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0.02 & 0.25 & 0 \\ 0 & 0.08 & 0.05 \end{bmatrix} \begin{bmatrix} 1000 \\ 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.02 \times 1000 + 0.25 \times 50 + 0 \\ 0 + 0.08 \times 50 + 0.5 \times 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 32.5 \\ 9 \end{bmatrix}$$

This is the number of surviving individuals after 1 year. Some of them grew up from the previous stage, some stayed in the same stage. Eggs are assumed to either die or grow into juveniles, so no eggs will survive and stay as eggs.

(c) We noticed that $RN_0 + SN_0 = AN_0 = (R+S)N_0$. In general, $(A+B)C = AC + BC$ for matrices A, B, and C.

Problem Set Solutions

1. Suppose a population of lake trout is growing according to the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

(a) What is the maximum possible growth rate for the population? When is it reached? (Write the answer in terms of r and/or K)

(b) After some investigation on K, a biologist decides to maximize his fishing yield by maintaining a population of lake trout at 1000 individuals. What is the value of K?

(c) After some more investigation on r, the biologist estimates that $r = 0.01$ individuals/(day$\cdot$individual). Now if 1200 additional lake trouts are added to the population, what will the instantaneous population growth rate be?

Solution:

(a) We calculate the maximum value of $\frac{dN}{dt}$:

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) = -\frac{r}{K}N^2 + rN \\ &= -\frac{r}{K}\left(N^2 - KN + \frac{K^2}{4}\right) + \frac{rK}{4} = -\frac{r}{K}\left(N - \frac{K}{2}\right)^2 + \frac{rK}{4}.\end{aligned}$$

Therefore, the maximum population growth rate is $\frac{rK}{4}$, obtained when $N = \frac{K}{2}$.

(b) Here is the underlying logic: The biologist wants to keep the number of trouts at 1000. When the population size increases, he takes away the excess trouts so that the population can stay at 1000. Therefore, in order to maximize the yield, the population should grow the fastest when there are 1000 trouts. From (a) we know that the maximum population growth rate is obtained when $N = \frac{K}{2}$, so

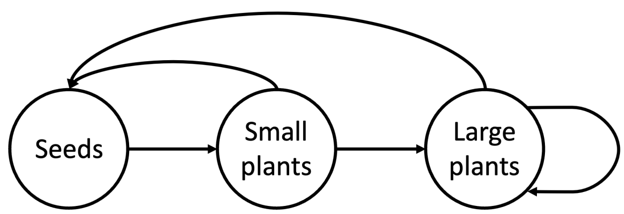
$$\frac{K}{2} = 1000$$

and thus the carrying capacity $K = 2000$.

(c) At the moment there are $1000+1200 = 2200$ trouts, so the instantaneous population growth rate is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = 0.01(2200)\left(1 - \frac{2200}{2000}\right) = -2.2\,\text{(individuals/day)}.$$

2. Suppose the life history of a plant is shown below:



We initially recorded 100 seeds, 250 small plants, and 50 large plants. After 1 year, we found that 20% of the seeds germinated and grew into small plants, 50% of small plants grew into large plants, and 80% of large plants survived. We also counted that each small plant can produce 5 seeds, while each large plant can produce 20 seeds on average.

(a) Write down the life table of the plant in a matrix A.

(b) Calculate the number of seeds, small plants, and large plants after 1 year.

(c) Calculate the number of seeds, small plants, and large plants after 2 years.

(d) Use an online calculating tool to calculate $A^n N_0$ for $n = 3, 4, 5, 6, 7, 8$. Then plot the natural logarithm (ln) number of individuals in each stage class versus time. What did you find?

Solution:

(a)

$$A = \begin{bmatrix} 0 & 5 & 20 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0.8 \end{bmatrix}, \; N_0 = \begin{bmatrix} 100 \\ 250 \\ 50 \end{bmatrix}.$$

(b)

$$AN_0 = \begin{bmatrix} 0 & 5 & 20 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} 100 \\ 250 \\ 50 \end{bmatrix} = \begin{bmatrix} 2250 \\ 20 \\ 165 \end{bmatrix}.$$

(c)

$$A^2N_0 = \begin{bmatrix} 0 & 5 & 20 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 100 \\ 250 \\ 50 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 20 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} 2250 \\ 20 \\ 165 \end{bmatrix} = \begin{bmatrix} 3400 \\ 450 \\ 142 \end{bmatrix}.$$

(d)

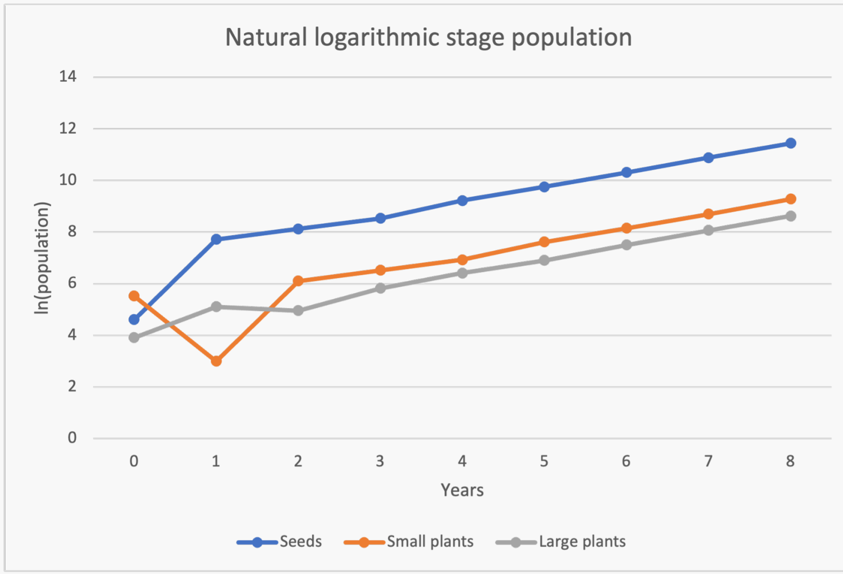
$$A^3N_0 = \begin{bmatrix} 5090 \\ 680 \\ 338.6 \end{bmatrix}, \; A^4N_0 = \begin{bmatrix} 10172 \\ 1018 \\ 610.88 \end{bmatrix}, \; A^5N_0 = \begin{bmatrix} 17307.6 \\ 2034.4 \\ 997.704 \end{bmatrix},$$

$$A^6N_0 = \begin{bmatrix} 30126.1 \\ 3461.52 \\ 1815.36 \end{bmatrix}, \; A^7N_0 = \begin{bmatrix} 53614.9 \\ 6025.22 \\ 3183.05 \end{bmatrix}, \; A^8N_0 = \begin{bmatrix} 93787.1 \\ 10723.0 \\ 5559.05 \end{bmatrix}.$$

We obtain the following chart:

Year	ln(Seeds)	ln(Small plants)	ln(Large plants)
0	4.605	5.521	3.912
1	7.719	2.996	5.106
2	8.132	6.109	4.956
3	8.535	6.522	5.825
4	9.227	6.926	6.415
5	9.759	7.618	6.905
6	10.313	8.149	7.504
7	10.890	8.704	8.066
8	11.449	9.280	8.623

We plot the natural logarithm stage population versus time as follows:



We notice that starting in Year 5, the logarithmic population increases nearly linearly for each stage, which means that the natural log growth rate is a constant (i.e. $\lambda = \ln(1+r_d)$). In other words, the population will eventually experience exponential growth! ($N(t) = (1 + r_d)^t N_0 = (e^\lambda)^t N_0 = N_0 e^{\lambda t}$)